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OPTICAL

INVESTIGATIONS.

CAUSTICS.

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PREFACE.

The Caustic of any proposed reflecting or refracting curve is the curve formed by the perpetual intersections of rays which have been reflected or refracted at the curve. Optically considered the determination of the equations of the caustics of the curves is more important than is commonly imagined, and is closely connected with the subject of optical images. If a luminous point be situated any where in the plane of a reflecting or refracting arc, and if an eye, situated in this plane, receive one of the reflected or refracted rays, it is sufficiently apparent that the luminous point will appear to the eye just mentioned to be situated somewhere in the direction of the ray which it receives. A question arises at what particular point of this indefinite line does the visible image of the luminous point appear, and in what way is this point to be determined for any given position of the luminous point, the eye, and the reflecting arc? This is a question to which theory cannot supply any answer. It is a result of *experiment* that the eye, looking as above described, along one of the reflected or refracted rays, will see the visible image of the luminous point at the particular point in which the said ray is a tangent to the caustic touched by every one of the reflected pencils. Thus, to take a very simple instance, if the luminous point be situated at the extremity of the diameter of a reflecting circle, it will be shewn that the caustic touched, in this case, by the reflected rays, is a cardioide; and this result, viewed with reference to its optical importance, exhibits to us as in a picture the varying position of the visible image of the luminous point corresponding to the varied position of the eye of the observer. From any given position of the eye, if we draw a tangent to the curve of the caustic we in fact, in constructing the point of contact, construct for the place of the visible image of the luminous point corresponding to the proposed situation of the eye.

Again, if a luminous point be placed before a plane refracting surface of a denser medium, it may be shewn that the caustic curve is the evolute of an ellipse; and the position and magnitude of the curve may be determined for any given position of the luminous point and the refracting surface.

If the eye of an observer be supposed to be situated in the plane of the caustic, and to receive one of the refracted rays, it follows, from the experimental principle before described, that the point in which the ray just mentioned touches the caustic curve, is the place of the visible image of the luminous point corresponding to the proposed place of the eye.

If in any proposed case, while the eye of the observer, and the position of the reflecting or refracting surface remain unchanged, the place of the luminous point is varied, it easily follows that the caustic curve will in general vary in position and magnitude, whilst its species is unaltered. At the same time, the place of the point which is the visible image of the luminous point, that is to say, the point in which a reflected or refracted ray, passing through the eye, touches the caustic, will in general be changed. And further if, on the same suppositions, the luminous point move in any given curve, it is plain that the said point of contact, and therefore the place of the visible image of the luminous point, will move in a certain curve. And the determination of the curve last mentioned, for any given form of the luminous curve, is, in fact, the problem of optical images in its most general form. The explanation of the mode of effecting the solution of this problem, would be foreign to the subject of the following pages; and the problem itself has been mentioned merely for the purpose of instancing the *optical* application of the forms of caustics.

It is the object of the present publication to determine, by a simple and satisfactory method, the equations of a variety of these curves; and it is believed that, independently of their physical applications, the simplicity and analytical elegance of some of the results, may excuse the space devoted to the examination of them.

OPTICAL INVESTIGATIONS.

CAUSTICS.

§. 1. In Fig. 1. *Q* is a luminous point from which proceed rays *QR*, *QS*, . . . that fall upon the concave reflecting arc *RSTU* . . and are reflected into the directions *Rq*, *Sr*, *Ts*, respectively, intersecting each other in the points *q*, *r*, *s*, . . The curve which passes through the points *q*, *r*, *s*, . . when these points approach each other indefinitely, that is to say, the curve formed by the perpetual intersections of the reflected rays, is termed a caustic.

§. 2. In order to find the equation of this curve referred to rectangular coordinates, let us investigate a method of determining, in general, the nature of the curve formed by the perpetual intersections of right lines, any one of which is defined by the equation

$$y = ax + \phi(a) \qquad (1)$$

in which $\phi(a)$ denotes any function of (a).

If we consider a second line of the system for which (a) becomes any other quantity

$$a + \delta a$$

it is plain that the equation of this line will be

$$y = \left\{a + \delta a\right\} . x + \phi \left\{a + \delta a\right\} \qquad (2)$$

The lines (1) and (2) intersect in a point whose coordinate (x) is determined by the equation

$$o = x \cdot \delta a + \phi(a + \delta a) - \phi(a) \qquad (3)$$

The equation (3) determines the abscissa (x) of the point in which any two lines of the system intersect. As the indeterminate quantity δa varies, the point of intersection of the lines defined by the equations (1) and (2), varies also; as δa converges to zero, the point of intersection above mentioned approaches constantly and indefinitely to coincidence with a point which, for the sake of distinction, we will call the point in which two *consecutive* lines of the system intersect. It is obvious that the abscissa (ξ) of this point will be given by the equation.

$$o = \xi \cdot da + d \cdot \phi(a).$$

It is essential to remark that the same value of the abscissa ξ would have been obtained by differentiating the equation (1) with respect to the arbitrary quantity (a), and by considering the variables y and x constant in the differentiation.

§. 3. In the equation

$$y = ax + b$$

representing any one of the proposed system of lines, if the constants (a) and (b) be supposed functions of a third quantity (c), so that

$$a = \phi(c) \;:\; b = \psi(c)$$

the equations of any two lines of the system will be

$$y = x \cdot \phi(c) + \psi(c)$$

$$y = x \cdot \phi(c + \delta c) + \psi(c + \delta c)$$

and the abscissa x of their point of intersection will be determined by the equation

$$o = x \left\{ \phi(c + \delta c) - \phi(c) \right\}$$
$$+ \psi(c + \delta c) - \psi(c)$$

or, by the equation,

$$o = x \cdot \frac{\phi(c + \delta c) - \phi(c)}{\delta c}$$
$$+ \frac{\psi(c + \delta c) - \psi(c)}{\delta c}$$

or, when the lines are consecutive, by the equation

$$o = \xi \cdot \frac{d\phi(c)}{dc} + \frac{d\psi(c)}{dc}$$

It follows, from the equation last written, that in the differentiation mentioned in the last page, we are at liberty to perform the differentiations with respect to any function whatever of the constants (a) and (b).

It is plain that the relation between the co-ordinates η and ξ of the caustic is to be obtained by eliminating (c) between the equations

$$\begin{aligned} \eta &= \xi\, \phi(c) + \psi(c) \\ o &= \xi \cdot \frac{d\phi(c)}{dc} + \frac{d\psi(c)}{dc} \end{aligned} \qquad (1)$$

§. 4. If the curve whose caustic is required can only be conveniently defined by a polar equation, it becomes necessary to adopt an analysis different from the preceding.

Let (u) be the length of the radius vector of any point in a right line drawn according to a certain law, and ω be the angle comprised between the radius vector and a fixed semiaxis having the same origin as (u). Further, let the perpendicular, whose length is (p), dropped from this point upon the right line considered, make with the fixed semiaxis an angle β, we shall have in every imaginable case the equation

$$u = \frac{p}{\cos.\,(\omega - \beta)} \qquad (1)$$

In this equation it is to be observed that the angles ω and β may be either positive or negative, and that they are supposed to have numerical values comprised between o and 180°. The positive angles are to be measured on one side of the semiaxis mentioned above, the negative angles on the other side.

In the case supposed we may make

$$p = F(\beta)$$

where F denotes any function, and may write instead of the equation (1), the following

$$u = \frac{F(\beta)}{\cos.\,(\omega - \beta)} \qquad (2)$$

representing any line of the proposed system of lines.

If we consider any other line of the system in which β becomes

$$\beta + \delta\beta$$

it is plain that the equation of this line will be

$$u = \frac{p + \delta p}{\cos.\ (\omega - \beta - \delta\beta)} \qquad (3)$$

where δp

represents the variation of (p) corresponding to the variation

$$\delta\beta$$

of β. The lines (2) and (3) will intersect in a point whose coordinate ω is given by the equation

$$o = \frac{p + \delta p}{\cos.\ (\omega - \beta - \delta\beta)} - \frac{p}{\cos.\ (\omega - \beta)}$$

or, by the equation,

$$o = p \left\{ \cos.\ (\omega - \beta) - \cos.\ (\omega - \beta - \delta\beta) \right\} + \delta p . \cos.\ (\omega - \beta)$$

or, by the equation,

$$o = \cos.\ (\omega - \beta) . \frac{\delta p}{\delta\beta} - p . \frac{\cos. \left\{ \omega - \beta - \delta\beta \right\} - \cos. \left\{ \omega - \beta \right\}}{\delta\beta}$$

or, when the lines are consecutive, by the equation

$$o = \cos.\ (\omega - \beta) . \frac{dp}{d\beta} - p . \sin.\ (\omega - \beta) \qquad (4)$$

§. 5. If we eliminate β between the equations (2) and (4), or between the following

$$u = \frac{F(\beta)}{\cos. (\omega - \beta)}$$

$$o = \frac{d\, F(\beta)}{d\, \beta} - F(\beta) \,.\, \tan. (\omega - \beta)$$

we shall obtain an equation between (u) and (ω) which will be the equation of the curve formed by the intersection of lines drawn according to the proposed law.

§. 6. The equations of the last article furnish us with a means of discovering, in certain cases, the relation between the radius vector and the perpendicular upon the tangent of the caustic curve with remarkable elegance and simplicity. The equations spoken of, or the following

$$u = \frac{p}{\cos. (\omega - \beta)}$$

$$o = \frac{d\, F(\beta)}{d\, \beta} - F(\beta) \,.\, \tan. (\omega - \beta)$$

give at once, by eliminating the angle $\omega - \beta$, the equation

$$\sqrt{u^2 - p^2} = \frac{d\, F(\beta)}{d\, \beta} \qquad (1)$$

between which equation, and the equation

$$p = F(\beta) \qquad (2)$$

if we eliminate β, be have the equation sought.

§. 7. As an example of the application of the formulæ of §. 3. let us determine the caustic produced by rays which proceeding parallel to the axis of (y) are incident upon a logarithmic curve *A P* (Fig. 2) whose equation referred to rectangular coordinates *O X*, *O A*, is

$$y = \epsilon^{x}$$

where ϵ is the base of Napier's system of logarithms. Let QP be one of the rays incident upon the curve at P and reflected into the direction PR intersecting the axis of (x) in R, and making with the normal IP an angle IPR (i) equal to the angle of incidence IPQ, If (b) and (c) be the coordinates of the point P, it is plain that the equation of the line PR will be

$$y - b = a.(x - c)$$

a being the tangent of the angle PRX. Now it is easily seen that

$$a = - \cot. 2i$$

that is to say

$$a = \frac{1}{2} \left\{ \tan. i - \cot. i \right\}$$

also we have

$$\tan. i = \frac{db}{dc} = \epsilon^{c}$$

$$\cot. i = \frac{dc}{db} = \epsilon^{-c}$$

so that the expression for (a) becomes

$$a = \frac{1}{2} \left\{ \epsilon^{c} - \epsilon^{-c} \right\}$$

If we adopt the notation of the article referred to, we shall evidently have

$$\phi(c) = \frac{1}{2} \left\{ \epsilon^{c} - \epsilon^{-c} \right\}$$

$$\psi(c) = \epsilon^{c} - \frac{c}{2} \left\{ \epsilon^{c} - \epsilon^{-c} \right\}$$

from which we find

$$\frac{d\phi(c)}{dc} = \frac{1}{2} \left\{ \epsilon^{c} + \epsilon^{-c} \right\}$$

$$\frac{d\,\psi(c)}{d\,c} = \overline{1 - c}\,.\,\frac{d\,\phi(c)}{d\,c}$$

These formulæ reduce the formulæ (1) of §. 3. to the following

$$o = 1 - c + \xi$$

$$\eta = \frac{1}{2}\left\{\epsilon^{o} + \epsilon^{-c}\right\}$$

from which we obtain, eliminating c, the equation

$$\eta = \frac{1}{2}\left\{\epsilon^{(1+\xi)} + \epsilon^{-(1+\xi)}\right\} \qquad (1)$$

which is that of the caustic.

In the figure, (Fig. 2.), ECD represents the caustic, that is to say, the catenary, which is the locus of the equation (1). In the figure referred to, C is the vertex of the caustic, and

$$OA = OB = BC = AC = 1, AD = \frac{1}{2}\left\{\epsilon + \frac{1}{\epsilon}\right\} \text{ etc.}$$

The caustic is formed by the perpetual intersection of the reflected rays produced backwards, and may be termed an *imaginary* caustic in contradistinction to *real* caustics, or those which are formed by the actual intersection of the reflected rays.

§. 8. As a second example of the application of the formulæ of §. 3. let us take the case in which rays parallel to its axis are incident upon the arc of a cycloid APC, whose axis is AB, and base DBC.

Let SP be one of the rays of light incident at P, and reflected into the direction PR meeting the axis of the curve in R. As before, let PI be the normal at the point

P, (i) the angle of incidence *SPI*, (a) the tangent of the angle *PRB*, we shall have

$$a = -\tan. 2i,$$

that is to say,

$$a = \frac{2\tan. i}{\tan.^2 i - 1} \qquad (1)$$

If (c) and (b) be respectively the abscissa and ordinate of the point P of the cycloid, we have evidently

$$\frac{db}{dc} = \sqrt{\frac{c}{2-c}}$$

if we suppose the radius of the generating circle of the cycloid to be equal to unity.

It is easily seen that

$$\cot. i = \frac{db}{dc} = \sqrt{\frac{c}{2-c}}$$

an equation by aid of which we easily reduce the equation (1) to the following

$$a = \frac{1}{1-c} \cdot (2c - c^2)^{\frac{1}{2}}$$

If then, we retain the notation of §. 3, we shall easily see that

$$\phi(c) = \frac{1}{1-c} \cdot (2c - c^2)^{\frac{1}{2}}$$

$$\psi(c) = b - \frac{c}{1-c} \cdot (2c - c^2)^{\frac{1}{2}}$$

whence we deduce, after reduction,

$$\frac{d\,\phi(c)}{d\,c} = + \frac{\overline{(2c - c^2)}^{\frac{1}{2}}}{(1 - c)^2}$$

$$\frac{d\,\psi(c)}{d\,c} = - \frac{(2c - c^2)^{\frac{1}{2}}}{(1 - c)^2}$$

These equations reduce the formulæ (1) of §. 3. to the following

$$o = (2c - c^2) - \xi$$

$$\eta = (2c - c^2)\,.\,\phi(c) + \psi(c)$$

whence we easily find

$$\xi = 2c - c^2$$

$$\eta = b + c\,.\,\sqrt{2c - c^2}$$

Thus we have

$$\frac{d\eta}{d\,c} = 2\,\sqrt{2c - c^2} = 2\,\sqrt{\xi}$$

$$\frac{d\,\xi}{d\,c} = 2(1 - c) = 2\,\sqrt{1 - \xi}$$

and consequently

$$\frac{d\eta}{d\xi} = \sqrt{\frac{\xi}{1 - \xi}} \qquad (2)$$

the differential equation of the caustic.

In the figure, (Fig. 3.), *CbB* represents the caustic, that is to say, the cycloid which is the locus of the equation (2). In the figure referred to, (*a*) is the vertex of the cycloid; *ab*, equal to the half of *AB*, its axis, and

B C its base, equal to the half of the base of the reflecting curve. *Pp* is one of the reflected rays touching the caustic in (p).

§. 9. I shall, in the next place, apply the formulæ of §. 5. to determine the nature of the caustic curve when rays proceeding from a luminous point situated in the pole of a logarithmic spiral, are incident upon the curve of the spiral.

Let S (Fig. 4.) be the pole of the spiral; SP any radius vector (r), and let the angle which it makes with a fixed semi-axis be called ϕ. Also let the relation between the quantities (r) and (ϕ) be given by the equation

$$r = A \,.\, \epsilon^{-\frac{\phi}{a}} \qquad (1)$$

where A and (a) are positive constants, and where ϵ is the base of Napier's system of logarithms.

If Fig. 4. represent the logarithmic spiral under consideration, and if SMX be the fixed axis from which the angle ϕ is measured, it is evident that SM is the value of SP when $\phi = o$, and consequently that $SM = A$. Further, (a) is the tangent of the acute angle (α), which the radius vector of the spiral makes at any point P, with the tangent at this point.

If we retain the notation of §. 4. we shall see that the equation of the indefinite right line PQ is

$$u = \frac{p}{\cos.\,(\omega - \beta)}$$

which becomes, for the particular point P,

$$r = \frac{p}{\cos.\,(\phi - \beta)} \qquad (2)$$

It is evident, from the simplest geometrical considerations, that we have, in every imaginable case,

$$\phi - \beta = 2\alpha - \frac{\pi}{2} \qquad (3)$$

where π is 180°. We easily obtain from the equations (2) and (3)

$$p = r \,.\, \sin.\, 2\alpha$$

or, restoring the value of r,

$$p = A \,.\, \epsilon^{-\frac{\phi}{a}} .\, \sin.\, 2\alpha$$

which becomes, in virtue of the equation (3),

$$p = A \,.\, \sin.\, 2\alpha \,.\, \epsilon^{-\frac{\beta + \zeta}{a}}$$

if we make

$$\zeta = 2\alpha - \frac{\pi}{2} \qquad (4)$$

By means of the value of (p), or $F(\beta)$, thus obtained, the last of the equations of §. 5, is reduced to the following, to wit,

$$o = \frac{1}{a} + \tan.\, (\omega - \beta)$$

whence we easily find

$$\omega - \beta = \alpha - \frac{\pi}{2} = \zeta - \alpha \qquad (5)$$

$$\cos.\, (\omega - \beta) = \sin.\, \alpha$$

and these formulæ reduce the equations of §. 5. to the following

$$u = A . \frac{\sin. 2\alpha}{\sin. \alpha} . \epsilon^{-\frac{\zeta + \beta}{a}}$$

$$\zeta + \beta = \omega + \alpha$$

from which we find, eliminating $\zeta + \beta$, the equation

$$u = A . \frac{\sin. 2\alpha}{\sin. \alpha} . \epsilon^{-\frac{\omega + \alpha}{a}} \qquad (6)$$

which is that of the caustic.

§. 10. The equation (6) evidently represents a logarithmic spiral, precisely similar and equal to the reflecting curve, but differently situated with respect to the fixed semiaxis SX. We can obtain a very simple and elegant construction for determining the position of any point (p) of the caustic, corresponding to a point P of the reflecting curve, by means of the formulæ of the preceding articles. Thus the first of the equations, (5), compared with the equation (3), gives at once

$$\phi - \omega = \alpha ; \phi = \omega + \alpha$$

whence it follows that, if P and p be two corresponding points in the reflecting curve and in the caustic, we shall have

$$SP = A . \epsilon^{-\frac{\phi}{a}},$$

$$Sp = A . \frac{\sin. 2\alpha}{\sin. \alpha} . \epsilon^{-\frac{\phi}{a}},$$

so that

$$\frac{SP}{Sp} = \frac{\sin. \alpha}{\sin. 2\alpha}.$$

which equation, combined with the equation

$$\angle\ PSp = \alpha$$

enables us to construct the point (p) when the point P is assigned.

In Fig. 5, pp represents the caustic whose equation we have just found, and to which the reflected rays Pp, Qp, are tangents in the points p. It is evidently wholly comprised within the reflecting spiral PQ.

§. 11. The ray SP, incident at the point P of the spiral, and reflected into the direction PQ, will in general be incident again upon the spiral at some point Q, and reflected into a certain direction QR. If we consider a ray incident at a point extremely near to the point P, it is evident that such a ray, after two reflexions at the curve, will intersect the reflected ray QR in some point q. We may inquire what the position of the point q is, when determined by the intersection of two *consecutive* rays; and further, what the nature of the curve is which passes through all such points as q. And this is, in other words, to inquire what is the nature of the caustic produced by rays proceeding from the pole of a logarithmic spiral, and incident twice upon the curve.

If we retain the notation of §. 5, we shall have, for the equation of the line QR, (Fig. 6.),

$$u = \frac{p}{\cos. (\omega - \beta)} \qquad (1)$$

Let us suppose

$$SP = r$$

$$SQ = s$$

Angle formed by (r) with $SX = \phi$

. . . . . (s) . . . $= \psi$

we easily find from the equation (1),

$$p = s \,.\, \cos.\, (\psi - \beta) \tag{2}$$

Let the tangents to the spiral, at the points P and Q, meet in T, it may be shewn, by the simplest geometrical considerations, that

$$\angle\, SQR = \angle\, PTQ = \pi - PSQ$$

whence we find

$$\cos.\, (\psi - \beta) = \sin.\, SQR = \sin.\, (\phi - \psi)$$

an equation which reduces the equation (2) to the following

$$p = s.\, \sin.\, (\phi - \psi)$$

or, putting for (s) its value,

$$p = A \sin.\, (\phi - \psi) \,.\, \epsilon^{-\frac{\psi}{a}}$$

which since

$$\psi - \beta = \phi - \psi - \frac{\pi}{2} \tag{3}$$

becomes

$$p = A \,.\, \sin.\, (\phi - \psi) \,.\, \epsilon^{-\frac{\zeta + \beta}{a}} \tag{4}$$

where ζ is now given by the equation

$$\zeta = \phi - \psi - \frac{\pi}{2} \tag{5}$$

If we examine the triangle ***PSQ*** we shall easily see that

$$r : s :: \sin.\left\{\pi - 2\alpha + \phi - \psi\right\} : \sin.\left\{\pi - 2\alpha\right\}$$

that is to say

$$\epsilon^{-\frac{\phi}{a}} : \epsilon^{-\frac{\psi}{a}} :: \sin.\left\{2\alpha - \overline{\phi - \psi}\right\} : \sin. 2\alpha$$

whence we deduce

$$\sin. 2\alpha = \epsilon^{\frac{\phi - \psi}{a}} . \sin.\left\{2\alpha - \overline{\phi - \psi}\right\} \qquad (6)$$

an equation which informs us that the angle $\phi - \psi$ is a constant angle.

The last of the equations of §. 5. becomes in this case

$$o = \frac{1}{a} + \tan.(\omega - \beta)$$

whence we easily find

$$\omega - \beta = \alpha - \frac{\pi}{2} : \cos.(\phi - \beta) = \sin. \alpha \qquad (7)$$

By the aid then of the equation (5) the formulæ of §. 5. are reducible to the following

$$u = A . \frac{\sin.(\phi - \psi)}{\sin. \alpha} . \epsilon^{-\frac{\beta + \zeta}{a}}$$

$$\zeta + \beta + \alpha = \phi - \psi + \omega$$

between which equations if we eliminate β we have the equation of the caustic, to wit,

$$u = A \frac{\sin.(\phi - \psi)}{\sin. \alpha} . \epsilon^{-\frac{(\omega + \phi - \psi - \alpha)}{a}}$$

representing a logarithmic spiral precisely similar and equal to the proposed logarithmic spiral but differently situated.

§. 12. The first of the equations (7) compared with the equation (3) gives us at once

$$\psi = \omega + \phi - \psi - \alpha$$

an equation from which we may easily obtain a construction similar to that in §. 10.

In Fig. 5. $q\,q$ represents the caustic whose equation we have last found and to which reflected rays $P\,q$, $Q\,q$ are tangents at the points q. It is easily seen that the curve $q\,q$ lies altogether between the caustic $p\,p$ and the reflecting curve.

§. 13. The ray $S\,P$ will evidently be reflected an indefinite number of times at the surface of the spiral, and it would be very easy by means of a method similar to the preceding, to shew that the caustic touched by rays reflected any number of times at the spiral is also a logarithmic spiral, similar and equal to the reflecting curve and differing from it only in position.

§. 14. Lastly I shall apply the formulæ of §. 6. to determine the relation between the radius vector and the perpendicular upon the tangent of the caustic curve in certain cases.

As a first example let the luminous point be situated at the extremity S of the diameter $S\,C\,A$ of the reflecting circle $S\,P'\,A$ (Fig. 7.) Let $S\,P'$ be one of the rays of light which, proceeding from S, is incident at P' upon the reflecting curve and is turned into the direction $P'\,P''$. Let the radius $C\,P'$ of the circle be called (a) and let (p) be the length of the perpendicular dropped from the centre C upon the reflected ray $P'\,P''$. Further let ϕ be the angle which the radius $C\,P'$ makes with the fixed semiaxis

CA. It is evident that we have, in every imaginable case

$$\phi - \beta = \frac{\pi}{2} - \frac{\phi}{2}$$

whence we deduce

$$\phi = \frac{2}{3}\left(\frac{\pi}{2} + \beta\right)$$

$$\cos.\,(\phi - \beta) = \sin.\,\frac{\phi}{2} = \sin.\,\frac{1}{3}\left(\frac{\pi}{2} + \beta\right)$$

Hence we have

$$F(\beta) = a\,.\,\sin.\,\frac{1}{3}\left(\frac{\pi}{2} + \beta\right)$$

and the equations (1) and (2) of §. 6. are thus reduced to the following;

$$\frac{p}{a} = \sin.\,\frac{1}{3}\left(\frac{\pi}{2} + \beta\right)$$

$$\frac{\sqrt{u^2 - p^2}}{\frac{a}{3}} = \cos.\,\frac{1}{3}\left(\frac{\pi}{2} + \beta\right)$$

whence we obtain, by squaring and adding,

$$\frac{p^2}{a^2} + \frac{u^2 - p^2}{\overline{\frac{a}{3}}\,\Big|^2} = 1 \quad .$$

§. 15. The ray $P' P''$ after reflexion at P' will be incident on the reflecting curve a second time, at some point P'', and reflected into a certain direction P'', P'''. It will again be reflected at the point P''', and turned, by reflexion, into the direction $P''' P''''$, and so on continually. In order to find the equation of the curve touched by rays which have been reflected any number (n) of times at the surface of the circle, let us, in addition to the denominations of the preceding article, call ψ the angle $P' S A$ between the first incident ray and the fixed semiaxis SA, and we shall evidently have, in every imaginable case, the formulæ

$$\phi - \beta = \frac{\pi}{2} - \psi$$

$$n \left\{ \pi - 2\psi \right\} = \pi - \phi$$

In these equations ψ is a certain angle comprised between zero and a right angle; ϕ, or the angle comprised between the radius $CP^{(n)}$ and the fixed axis CA, is a positive angle in the cases in which every one of the points

$$P^{(1)}, P^{(2)}, \; . \; . \; . \; . \; P^{(n)}$$

is on the same side of the axis SCA: it is negative in all other cases, and is equal, as to its numerical value, to the arc traced out by a point which, setting out from the point A, and moving in the direction of the ray of light, describes the arc comprised between the points A and $P^{(n)}$. In the same equation, β is the angle comprised between the semiaxis CA and the perpendicular $p^{(n)}$, dropped from C upon the ray reflected at the point $P^{(n)}$: it is a positive angle in the cases in which, ϕ being positive, the perpendicular before mentioned falls upon the same side of the diameter SCA with the first point $P^{(1)}$ of reflexion: it is negative in every other case, and is equal, as to its numerical value, to the arc traced out by a point which, setting out from the point A, and moving in the direction of the ray of light, describes the arc comprised between the point *A* and the point where the perpendicular $p^{(n)}$ produced meets the circumference of the reflecting circle.

From the equations of the last page we easily obtain

$$\psi = \frac{\overline{n-1} \,.\, \pi + \phi}{2n}$$

$$\frac{\phi}{2n} = \frac{1}{2n+1}\left\{\beta + \frac{\pi}{2n}\right\}$$

and consequently

$$\cos.\,(\phi - \beta) = \sin.\,\psi = \sin.\left\{\frac{\beta}{2n+1} + \frac{2n-1}{2n+1} \cdot \frac{\pi}{2}\right\}$$

thus the equations of §. 6. become

$$\frac{p}{a} = \sin.\left\{\frac{\beta}{2n+1} + \frac{2n-1}{2n+1} \cdot \frac{\pi}{2}\right\}$$

$$\frac{\sqrt{u^2 - p^2}}{\left(\frac{a}{2n+1}\right)} = \cos.\left\{\frac{\beta}{2n+1} + \frac{2n-1}{2n+1} \cdot \frac{\pi}{2}\right\}$$

whence we find, by squaring and adding,

$$\frac{p^2}{a^2} + \frac{u^2 - p^2}{\left(\frac{a}{2n+1}\right)^2} = 1$$

§. 16. It may easily be shewn that the equation of an epycycloid, referred to the centre of the *base* of the epycycloid, is

$$\left(\frac{p}{c+2b}\right)^2 + \frac{u^2-p^2}{c^2} = 1$$

in which equation c is the radius of the base, and b the radius of the revolving circle. If we compare this equation with the equation at the end of the last article, we shall easily see that the curve touched by the rays which have been reflected (n) times at the circle, is an epycycloid described by a point in the circumference of a circle whose radius is

$$\frac{n}{2n+1} \cdot a$$

which revolves on another circle concentric with the reflecting circle, and having a radius equal to

$$\frac{1}{2n+1} \cdot a$$

§. 17. The equation obtained at the end of §. 15. determines the nature of the caustic curve, but fails to define its position. It is plain that the position of the epycycloid represented by the equation referred to, will also be known, if we can by any means ascertain the position of the axis of the curve, that is to say, of the diameter of the circle which cuts the curve at a right angle. In order to this we must evidently find the value of the angle β corresponding to the point of the caustic for which the radius vector and the perpendicular upon the tangent are the same.

If we refer to the equations of §. 15. we shall easily see that the value of the angle β, corresponding to the point under consideration, must be given by the equations

$$o = \cos. \left\{\frac{\beta}{2n+1} + \frac{2n-1}{2n+1} \cdot \frac{\pi}{2}\right\}$$

$$1 = \sin. \left\{ \frac{\beta}{2n+1} + \frac{2n-1}{2n+1} \cdot \frac{\pi}{2} \right\}$$

whence we find

$$\frac{\beta}{2n+1} + \frac{2n-1}{2n+1} \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

and therefore

$$\beta = \pi$$

This equation shews evidently that the axis of the caustic, touched by rays which have been reflected any number of times at the surface of the circle, coincides with the diameter *S C A* passing through the luminous point *S*.

It is evident that the equation just obtained serves to determine completely the position of the caustic curve with respect to the axis *C A* of the reflecting circle. In fact, the equation referred to shews that the caustic, after any number (n) of reflexions, cuts the axis *A C* perpendicularly at its extremity *S*. Hence it follows, that in the position just mentioned, the generating point of the caustic must be situated at the extremity of that diameter of the revolving circle which, produced, passes through the centre *C* of the base. In order, then, to trace the caustic in the general case, we have only to describe a circle with centre *C* and radius

$$\frac{a}{2n+1}$$

meeting the axis *C S* in some point *P*, to bisect the distance *P S* in *D*, and with centre *D*, and radius *D P*, to describe the circle *S P*, and to conceive the latter circle to revolve upon the former. A point *P* which, in the beginning of the motion, coincided with *S*, will evidently describe the caustic.

Since, in the general case, the radius of the revolving circle is equal to (*n*) times the radius of the base, it is evident that the first of these circles will have completed $\frac{n}{2}$ revolutions about the second, when the generating point *P* meets the last-named circle. If (*n*) be an even number, it is plain that the point where the caustic meets the base, is situated in the line *CS*, and between the points *C* and *S;* if (*n*) be an odd number, this point lies in the line *CA*.

Figures 8, 9, 10, represent, respectively, the caustics touched by rays which have been reflected, once, twice, and three times, at the surface of the reflecting circle.

§. 18. I shall, in the next place, apply the formulæ of §. 6. to determine the equation of the caustic when parallel rays are incident upon a circular reflecting arc *AP′*, as in Fig. 11. In the said figure let us suppose *SP′* to be one of the system of parallel rays incident upon the curve at *P′*, and turned, by reflexion, into the direction *P′ P″*, and let us retain the denominations of the preceding problem, that is to say, let ϕ be the angle formed by the radius *CP′* with the fixed semiaxis *CA*, p the perpendicular dropped from the centre *C* upon the reflected ray *P′P″*, β the angle comprised between the said perpendicular and the axis before mentioned. It is evident that we have, in every imaginable case,

$$\phi - \beta = \frac{\pi}{2} - \phi$$

whence we deduce

$$\phi = \frac{1}{2}\left\{\frac{\pi}{2} + \beta\right\}$$

$$\cos. (\phi - \beta) = \sin. \phi = \sin. \frac{1}{2}\left(\beta + \frac{\pi}{2}\right)$$

so that the equations so often referred to in §. 6. become, in this case,

$$\frac{p}{a} = \sin. \frac{1}{2}\left(\beta + \frac{\pi}{2}\right)$$

$$\frac{\sqrt{u^2 - p^2}}{\frac{a}{2}} = \cos. \frac{1}{2}\left(\beta + \frac{\pi}{2}\right)$$

whence we find,

$$\frac{p^2}{a^2} + \frac{u^2 - p^2}{\left(\frac{a}{2}\right)^2} = 1$$

the equation of the caustic in this case.

§. 19. As in the last problem we may observe that the ray $P' P''$ will be incident a second time on the reflecting curve at some point P'' and reflected into a certain direction $P'' P'''$. It will again be reflected at the point P''' and turned by reflexion into a certain direction $P''' P''''$, and so on continually. In order to find the equation of the curve touched by rays which have been reflected any number (n) of times at the surface of the circle, let us, in addition to the denominations of the preceding article, call θ the angle $P' C A$ comprised between the radius passing through the first point of incidence and the fixed semiaxis, and we shall evidently have, in every imaginable case, the equations

$$\phi - \beta = \frac{\pi}{2} - \theta$$

$$\theta - \phi = \overline{n-1} \, . \, (\pi - 2\theta)$$

in which equations θ is a certain angle comprised between zero and a right angle and in which ϕ and β are supposed to be measured after the manner described in the corresponding case of §. 15.

From the equations just written we readily obtain

$$\theta = \frac{\overline{n-1} \, . \, \pi + \phi}{2n - 1}$$

$$\frac{\phi}{2n-1} = \frac{1}{2n} \left\{ \beta + \frac{1}{2n-1} \cdot \frac{\pi}{2} \right\}$$

and consequently

$$\cos. \, (\phi - \beta) = \sin. \, \theta = \sin. \left\{ \frac{\beta}{2n} + \frac{2n-1}{2n} \cdot \frac{\pi}{2} \right\}$$

Thus the equations of §. 6. become

$$\frac{p}{a} = \sin. \left\{ \frac{\beta}{2n} + \frac{2n-1}{2n} \cdot \frac{\pi}{2} \right\}$$

$$\frac{\sqrt{u^2 - p^2}}{\left(\frac{a}{2n}\right)} = \cos. \left\{ \frac{\beta}{2n} + \frac{2n-1}{2n} \cdot \frac{\pi}{2} \right\}$$

whence we find for the equation of the caustic

$$\frac{p^2}{a^2}+\frac{u^2-p^2}{\left(\frac{a}{2n}\right)^2}=1$$

§. 20. By referring to §. 15. it is easily seen that the curve touched by rays which have been reflected (n) times at the surface of the circle represented in Fig. 12. is an epycycloid described by a point in the circumference of a circle whose radius is

$$\left\{1-\frac{1}{2n}\right\}\cdot\frac{a}{2}$$

which revolves on another circle concentric with the reflecting circle and having a radius equal to

$$\frac{1}{n}\cdot\frac{a}{2}$$

Further it is evident that the radius vector which meets the epycycloid at a right angle will make with the fixed semiaxis an angle β such that

$$o=\cos.\left\{\frac{\beta}{2n}+\frac{2n-1}{2n}\cdot\frac{\pi}{2}\right\}$$

$$1=\sin.\left\{\frac{\beta}{2n}+\frac{2n-1}{2n}\cdot\frac{\pi}{2}\right\}$$

from which we find

$$\beta=\frac{\pi}{2}$$

an equation which informs us that the radius vector sought is perpendicular to the diameter *A C.*

As in §. 17. it is evident that the equation (1) will enable us to determine completely the position of the epycycloid which is the locus of the equation at the end of §. 19. Draw CDP perpendicular to AS (Fig. 13.;) with centre C and radius

$$\frac{1}{n}\cdot\frac{a}{2}$$

let a circle be described meeting CP in E: bisect EP in D; and with centre D and radius DP describe the circle PE: it is evident that, if the circle PE revolve on the base EP the caustic in the present case will be traced out by a point P which, in the beginning of the motion, coincided with the extremity of the radius CEP.

Since, in the general case, the radius of the revolving circle is equal to $\left(n-\frac{1}{2}\right)$ times the radius of the base, it is evident that the generating point P will meet the circumference of the latter circle for the first time at a point P distant from the point E by an arc equal to the $\left(\frac{n}{2}-\frac{1}{4}\right)$th. part of the whole circumference EP: for the second time at a point P distant from E by three times the above-mentioned arc, etc.

Figs. 12. and 13. represent respectively the caustics touched by the rays which have been reflected once and twice at the surface of the circle.

§. 21. I shall now take one or two of the cases in which the surface, on which the rays are incident is a refracting surface, and as a first example I shall take the case in which parallel rays are incident upon the refracting circle AP (Fig. 14.) Let the radius CP of this circle be called (a) and let (p) be the length of the perpendicular CY dropped from the centre C upon the direction PY of the refracted ray *PR*. Further, in addition to the denomination of the problem of §. 18. let us call ψ the angle of refraction *CPY* comprised between the radius *CP* and the refracted ray *PY*: lastly, let us suppose the refracting surface *AP* less dense than the ambient medium and let us put (m) for the ratio of the sine of incidence to the sine of refraction out of the said surface into the ambient medium. It is evident that we shall have, in every imaginable case, the equation

$$\beta - \phi = \frac{\pi}{2} - \psi$$

whence we deduce

$$\beta = \frac{\pi}{2} + \phi - \psi \tag{1}$$

also we have, by the law of refraction,

$$\frac{\sin.\ \phi}{\sin.\ \psi} = m$$

whence we find

$$\phi = \overset{-1}{\sin.} \left\{ m \sin.\ \psi \right\}$$

an equation which reduces the equation (1) to the following

$$\beta = \frac{\pi}{2} - \psi + \overset{-1}{\sin.} \left\{ m \sin.\ \psi \right\}$$

which gives, by differentiation,

$$1 = \left\{ \frac{m \,.\, \cos.\, \psi}{\sqrt{\left\{ 1 - (m \sin.\, \psi)^2 \right\}}} - 1 \right\} . \frac{d\psi}{d\beta} \qquad (2)$$

also we have

$$p = a \,.\, \sin.\, \psi \qquad (3)$$

by means of which formulæ we readily reduce the equation (2) to the following

$$1 = \left\{ m \,.\, \sqrt{\frac{a^2 - p^2}{a^2 - (mp)^2}} - 1 \right\} \frac{d\psi}{d\beta} \qquad (4)$$

The same formulæ gives, by differentiation,

$$\frac{dp}{d\psi} = \sqrt{a^2 - p^2}$$

which combined with the equation (4) gives the equation

$$\sqrt{a^2 - p^2} = \left\{ m \,.\, \sqrt{\frac{a^2 - p^2}{a^2 - (mp)^2}} - 1 \right\} . \frac{dp}{d\psi} . \frac{d\psi}{d\beta}$$

that is to say the equation

$$\sqrt{a^2 - p^2} = \left\{ m \,.\, \sqrt{\frac{a^2 - p^2}{a^2 - (mp)^2}} - 1 \right\} . \frac{dp}{d\beta}$$

and finally, in virtue of the formulæ of §. 6. and by reduction, the equation

$$1 = m\sqrt{\frac{u^2 - p^2}{a^2 - (mp)^2}} - \sqrt{\frac{u^2 - p^2}{a^2 - p^2}} \qquad (5)$$

which is that of the caustic.

§. 22. In the case in which the surface is a refracting circle of a rarer medium and the luminous point situated at the extremity S (Fig. 15) of a diameter of the curve we may easily shew, by a method similar to the preceding, that the equation of the caustic curve is

$$1 = 2\mu\sqrt{\frac{u^2 - p^2}{a^2 - (mp)^2}} - \sqrt{\frac{u^2 - p^2}{a^2 - p^2}}$$

§. 23. If in the two last articles we make

$$\mu = -1$$

it is easily seen that we return to the case in which the refracting surfaces become reflectors. On this supposition the equations at the end of §. 22. and §. 21. become respectively, after reduction,

$$\frac{p^2}{a^2} + \frac{u^2 - p^2}{\left(\frac{a}{3}\right)^2} = 1$$

and
$$\frac{p^2}{a^2} + \frac{u^2 - p^2}{\left(\frac{a}{2}\right)^2} = 1$$

which are the equations obtained in §. 14. and §. 18. respectively.

§. 24. The equations of §. 21. will enable us with ease to trace the curve whose equation was found in the article referred to. This case is illustrated by Fig. 16. in which *BAD* is a refracting circular arc, subtending, at its centre *C*, an angle *BCD* ($2a$), and in which *ppFpp* represents the caustic touched by the refracted rays.

It is evident that when $\psi = o$ we must have $p = o$, and therefore from the equation (4)

$$u = \frac{a}{m - 1}$$

so that the caustic meets the axis *AC* produced in a point *F* distant from *C* by the quantity just put down.

If we take the extreme ray *SB*, incident at *B*, and refracted into the direction *BR*, we shall have for this particular ray, by the equations of the article referred to,

$$\phi = \alpha : \text{sin.}\ \phi = m\ .\ \text{sin.}\ \psi : p = a\ \text{sin.}\ \psi$$

$$1 = \mu \sqrt{\frac{u^2 - p^2}{a^2 - p^2}} - \sqrt{\frac{u^2 - p^2}{a^2 - p^2}}$$

equations which, for any assumed values of (α) and (μ), will determine the point (p) which limits the caustic.

The curve will be perpendicular to the radius vector (u) when $u = p$, that is to say, in virtue of the equation last written, when

$$o = a^2 - (mp)^2, \text{ or } p = \frac{a}{m}$$

For this particular position we have evidently

$$\psi = \sin.^{-1}\left(\frac{1}{m}\right) ; \phi = \sin.^{-1}\left\{ m \sin. \psi \right\} = \frac{\pi}{2}$$

equations which shew us that if SD were a ray incident upon the reflecting curve at a right angle, DT the refracted ray, and Cp a perpendicular dropped from the centre of the circle upon the direction of the said refracted ray, the point of intersection (p) of the lines just mentioned is the point of the caustic at which the curve is perpendicular to the radius vector.

Also the length of this radius vector, and the angle (β) which it forms with the fixed semiaxis CA, are given by the equations

$$u = \frac{a}{m}$$

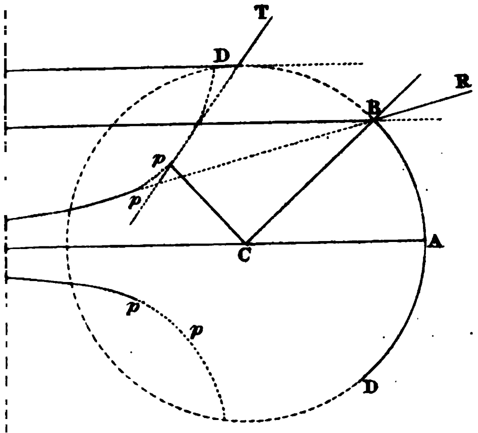
$$\beta = \pi - \sin.^{-1}\left\{\frac{1}{m}\right\}$$

In the figure the darkened line pFp represents the portion of the caustic which in the present case is actually touched by the refracted rays produced backwards: the dotted lines, on the contrary, represent the portions which may be considered as due to the analytical mode of treating the subject.

§. 25. In a similar manner we may determine the form of the caustic which is defined by the equation at the end of §. 22. see Fig. 17.

§. 26. In the cases in which the actual equation of the caustic curve cannot be deduced by any of the above methods, the formulæ of §. 5. and §. 6. will be found very convenient for the purpose of constructing the curve by points.

§. 27. The examples which have been selected in the preceding pages, as illustrative of the formulæ of the first six articles, are by no means the most intricate to which they may be applied. They are recommended, as it would seem, by their simplicity and elegance, and moreover involve a considerable number of results which are not to be found elsewhere, or which, at any rate, are not generally known. The method of discovering the relation between the radius vector and the perpendicular upon the tangent of the caustic curve, may be employed with advantage in the solution of a variety of problems in which the object is to determine the nature of curves formed by the intersections of lines drawn according to a given law. In fact it is plain that, in the investigations of the formulæ of the first six articles, no supposition was made restricting them to the subject of caustics. These applications are, however, foreign to the subject of the present publication, and are, besides, so simple and obvious, as to present little difficulty to the student.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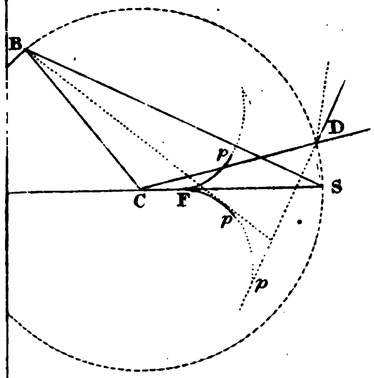
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